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to my wife, Natasha Zabzina

Generalized Kähler geometry, gerbes, and all that

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Abstract

This work is based on the talk delivered at Poisson 2008. We review the recent advances in Generalized Kähler geometry while stressing the use of Poisson and symplectic geometry. The derivation of the generalized Kähler potential is sketched and the relevant global issues are discussed.

1 Introduction

Kähler geometry plays a prominent role in mathematical physics. In particular, it is quite important in modern string theory. The two dimensional supersymmetric $N = (2,2)$ sigma model should have a Kähler target. The corresponding quantum theory should be defined over a Calabi-Yau manifold. Over the last two decades the study of these supersymmetric sigma models and their different relatives led to advances in such topics as mirror symmerty, Gromov-Witten invariants and topological strings.

However, in 1984 it was pointed out by Gates, Hull and Roček [2] that the sigma models with a Kähler target are not the most general supersymmetric $N = (2,2)$ model. They found that the target manifold for these general models should correspond to bihermitian geometry together with some integrability conditions. The interest in this type of the geometry has been revived after 2002 when Hitchin introduced the notion of generalized complex structure [5]. In [3] Gualtieri gave the alternative description of the Gates-Hull-Roček geometry within the framework of generalized complex geometry and he suggested a new name, generalized Kähler geometry. Indeed from the point of view of physics this is a very natural name. There is hope that many ideas and concepts can be extended to this generalized framework.

In this contribution our goal is modest and we would like to discuss the different geometrical features of generalized Kähler geometry. We would especially like to stress the Poisson and symplectic aspects of this geometry. Our intention will be to review and summarize a number of works [9, 10, 11, 8] written over a few last years. All of these works were inspired by the tools of supersymmetric sigma models. Here we provide the geometrical summary without any reference to sigma models.

The contribution is organized as follows: In Section 2 we review the standard facts about Kähler geometry. Section 3 contains the definition and basic properties of generalized Kähler geometry. In Section 4 we explore the different local description of the geometry and introduce the notion of a generalized Kähler potential. Section 5 deals with ways of gluing the local description and with the interpretation in terms of gerbes. Section 6 presents the summary and a list of open questions.

2 Kähler geometry

Let us remind the reader of a few well-known facts about Kähler geometry. In particular we want to discuss the local description of the geometry and the way of gluing together the local data into a global object.

Consider a complex manifold with a Hermitian metric (M, J, g). The manifold M is

called Kähler if the two-form $\omega = gJ$ is closed, $d\omega = 0$. The corresponding metric g is called a Kähler metric. The Kähler metrics come in infinite families since on M we can define a new closed 2-form

$$\omega' = \omega + i\partial\bar{\partial}\phi \ ,$$

which defines another Kähler metric g' provided that ϕ is a sufficiently small function. The positivity of the metric is an open condition and thus can be preserved under small deformations.

Choose an open cover $\{U_\alpha\}$ of M where all open sets and intersections are contractible. Since ω is a closed $(1,1)$-form then locally on the patch U_α we can write

$$\omega = i\partial\bar{\partial}K_\alpha \ , \tag{2.1}$$

where $K_\alpha(z, \bar{z})$ is a real function on U_α which should gives rise to a positive metric. Such a function K_α is called a Kähler potential. Thus locally, provided we choose the complex coordinates $(z, \bar{z})$, the Kähler geometry is defined by any real function which gives rise to a positive metric.

Assume that $\omega/2\pi \in H^2(M, \mathbb{Z})$. The way to glue the formula (2.1) on the intersection $U_\alpha \cap U_\beta$ is

$$K_\alpha - K_\beta = F_{\alpha\beta}(z) + \bar{F}_{\alpha\beta}(\bar{z}) \ , \tag{2.2}$$

where $F_{\alpha\beta}(z)$ is a holomorphic function on $U_\alpha \cap U_\beta$. Using the fact that ω is an integral 2-form we can define the holomorphic transition functions

$$G_{\alpha\beta}(z) = e^{F_{\alpha\beta}(z)} \ : \ U_\alpha \cap U_\beta \ \rightarrow \ \mathbb{C}_* \ ,$$

which satisfy the cocycle condition and the Hermiticity condition

$$G_{\alpha\beta}G_{\beta\gamma}G_{\gamma\alpha} = 1 \ , \qquad G_{\alpha\beta}\bar{G}_{\alpha\beta} = e^{K_\alpha}e^{-K_\beta} \ . \tag{2.3}$$

Therefore we are dealing with a holomorphic line bundle with Hermitian structure. The Kähler potential can be defined as

$$K_\alpha = \log ||s_\alpha||^2 \ , \tag{2.4}$$

where s_α is a local section of a holomorphic line bundle and $||s_\alpha||$ is defined through the Hermitian metric on the line bundle. The Kähler form $\omega/2\pi$ is the first Chern class of this holomorphic line bundle with Hermitian structure. This gives us both a local and a global description of the Kähler geometry.

3 Generalized Kähler manifolds

Generalized Kähler geometry $(M, J_\pm, g, H)$ was introduced originally in [2] as a target manifold for the general $N = (2, 2)$ supersymmetric sigma models. The geometry was specified by two complex structures $J_\pm$, a bihermitian metric g and a closed 3-form H with the following conditions satisfied

$$\nabla^\pm J_\pm = 0 , \qquad \nabla^\pm = \nabla \pm g^{-1} H . \tag{3.5}$$

Equivalently, the generalized Kähler geometry can be defined as a bihermitian manifold $(M, J_\pm, g)$ satisfying the following integrability conditions

$$d_+^c \omega_+ + d_-^c \omega_- = 0 , \qquad d d_\pm^c \omega_\pm = 0 , \tag{3.6}$$

where $\omega_\pm = g J_\pm$ and $d^c = i(\bar{\partial} - \partial)$ with the subscripts "$\pm$" referring to the $J_\pm$ complex structures. The closed 3-form H is

$$H = d_+^c \omega_+ = -d_-^c \omega_- . \tag{3.7}$$

The special case $J_+ = J_-$ coincides with the definition of the Kähler manifold. The generalized complex description of this bihermitian geometry was given by Gualtieri in [3]. In the generalized complex language the name "generalized Kähler geometry" appears very naturally. In what follows we will not use the language of the generalized geometry, although it appears to be very useful for the discussion of some of the issues.

The questions we would like to ask are the following: Can we generalize the simple description of Kähler geometry reviewed in Section 2 to the generalized Kähler case? Namely, can we describe the local geometry in terms of a single real function (potential)? If yes, how do we glue them together? In the rest of the contribution we will try to answer these questions.

The definition of generalized Kähler geometry can be stated in many different, but equivalent ways. For example, the first condition in (3.6) can be reformulated by saying that the bivectors

$$\pi_\pm = (J_+ \pm J_-) g^{-1} \tag{3.8}$$

are Poisson structures [12]. The Schouten bracket between two Poisson structures defines H as follows

$$[\pi_+, \pi_-]_s = -4 g^{-3} H . \tag{3.9}$$

Moreover it has been observed in [6] that the bivector

$$\sigma = [J_+, J_-] g^{-1} \tag{3.10}$$

is the real (imaginary) part of the holomorphic Poisson structure with respect to both complex structures. Namely we define $\sigma_\pm = J_\pm \sigma$ to be the imaginary (real) part of these holomorphic Poisson structures. The complex bivector $(\sigma - i\sigma_\pm)$ is a type $(2,0)$ holomorphic bivector for $J_\pm$ complex structure and it is Poisson (Schouten nilpotent). This implies that σ and $\sigma_\pm$ are a pair of real compatible Poisson structures. Obviously σ, $\sigma_\pm$ have the same symplectic leaves, although they define different symplectic structures on the leaf. The holomorphic Poisson structures described above are $(2,0)+(0,2)$ parts of the real Poisson structures $\pi_\pm$ [6]. Thus for the J_+ complex structure we have

$$\pi_\pm^{(2,0)} + \pi_\pm^{(0,2)} = \mp \frac{1}{2} J_+ \sigma$$

and likewise for the J_- complex structure we have

$$\pi_\pm^{(2,0)} + \pi_\pm^{(0,2)} = \frac{1}{2} J_- \sigma \ .$$

Thus we see that there are quite a few Poisson structures on generalized Kähler manifold. Indeed their presence is crucial for the local analysis of the geometry.

4 Local description

In the previous Section we have described two real Poisson structures $\pi_\pm$ and the real part σ of the holomorphic Poisson structure. It is important to stress that π_+ and π_- do not have any common Casimir functions. Moreover the leaf of σ is always inside of the leaves for $\pi_\pm$. Indeed the leaves of π_+ and π_- intersect only along a leaf of σ.

Consider a neighborhood of a regular point of a generalized Kähler manifold (i.e., there exists a neighborhood of the point where the ranks of $\pi_\pm$ are constant). We can choose the coordinates adapted to the symplectic foliations of the different Poisson structures $\pi_\pm$, σ and complex structures $J_\pm$. Namely we can choose the complex coordinates for J_+

$$(z, \bar{z}, z', \bar{z}', x_+, \bar{x}_+) \ , \tag{4.11}$$

such that $(x_+, \bar{x}_+)$ are the coordinates along the leaf of σ, $(z', \bar{z}', x_+, \bar{x}_+)$ are the coordinates along the leaf of π_- and $(z, \bar{z}, x_+, \bar{x}_+)$ are the coordinates along the leaf of π_+. Analogously we can choose J_- complex coordinates

$$(z, \bar{z}, z', \bar{z}', x_-, \bar{x}_-) \ , \tag{4.12}$$

such that $(x_-, \bar{x}_-)$ are the coordinates along the leaf of σ, $(z', \bar{z}', x_-, \bar{x}_-)$ are the coordinates along the leaf of π_- and $(z, \bar{z}, x_-, \bar{x}_-)$ are the coordinates along the leaf of π_+. For these two

choices we can pick up the same coordinates along kernels of π_- and π_+. The possibility of choosing these coordinates follows from the general properties of the Poisson geometry and the definitions (3.8), (3.10) of Poisson structures in terms of the complex structures. The crucial fact is that these two sets of the coordinates are related to each other by the Poisson diffeomorphism for σ, i.e. the diffeomorphism preserving σ.

4.1 $\sigma = 0$

We start by considering the special case of generalized Kähler geometry when $\sigma = 0$ or equivalently, two complex structures commute $[J_+, J_-] = 0$. There exists the integrable local product structure $\Pi = J_+ J_-$ which gives rise to the real polarization. We can introduce four differentials: ∂_z, $\partial_{z'}$ and their complex conjugate $\bar{\partial}_{\bar{z}}$, $\bar{\partial}_{\bar{z}'}$. All these differential anticommute with each other. The standard differential we were using before can be written as follows

$$d = \partial_z + \partial_{z'} + \bar{\partial}_{\bar{z}} + \bar{\partial}_{\bar{z}'} \ , \qquad d^c_+ = -i\partial_z - i\partial_{z'} + i\bar{\partial}_{\bar{z}} + i\bar{\partial}_{\bar{z}'} \ , \qquad d^c_- = -i\partial_z + i\partial_{z'} + i\bar{\partial}_{\bar{z}} - i\bar{\partial}_{\bar{z}'} \ .$$

The corresponding generalized Kähler metrics come in infinite families. Namely 2-forms $\omega'_\pm$ on $(M, J_\pm, g)$

$$\omega'_+ = \omega_+ + i(\partial_z\bar{\partial}_{\bar{z}} - \partial_{z'}\bar{\partial}_{\bar{z}'})\phi \ , \tag{4.13}$$

$$\omega'_- = \omega_- + i(\partial_z\bar{\partial}_{\bar{z}} + \partial_{z'}\bar{\partial}_{\bar{z}'})\phi \ , \tag{4.14}$$

satisfy the condition (3.6) if the forms $\omega_\pm$ satisfy the same condition. The forms $\omega'_\pm$ define a new bihermitian metric if ϕ is small enough.

Locally on a patch U_α we can solve the conditions (3.6) as follows

$$\omega_\pm = i(\partial_z\bar{\partial}_{\bar{z}} \mp \partial_{z'}\bar{\partial}_{\bar{z}'})K_\alpha \ , \tag{4.15}$$

where $K_\alpha(z, z', \bar{z}, \bar{z}')$ is a real function such that the corresponding bihermitian metric is positive. Accordingly, as result of (3.7) the 3-form is given

$$H = (\partial_z\bar{\partial}_{\bar{z}'}\bar{\partial}_{\bar{z}} + \partial_{z'}\partial_z\bar{\partial}_{\bar{z}} + \bar{\partial}_{\bar{z}}\partial_{z'}\bar{\partial}_{\bar{z}'} + \partial_{z'}\partial_z\bar{\partial}_{\bar{z}'})K_\alpha \ . \tag{4.16}$$

This type of generalized Kähler geometry is linear generalization of the Kähler case. Indeed we are dealing with the local product of two Kähler geometries.

4.2 invertible σ

Now let us consider another special type of generalized Kähler geometry when σ is invertible. Thus $\Omega = \sigma^{-1}$ is a symplectic structure which is the real part of the holomorphic

symplectic structure. Their imaginary parts are given by the corresponding symplectic structures $\Omega_\pm = \Omega J_\pm$. Thus $(\Omega + i\Omega_\pm)$ are the holomorphic symplectic structures for $J_\pm$. The symplectic forms Ω, $\Omega_\pm$ encode whole geometry and we can read off from them the complex structures $J_\pm$ and the bihermitian metric.

The crucial property of a holomorphic symplectic structure is that the complex and Darboux coordinates can be chosen simultaneously. Thus locally we can pick up the Darboux complex coordinates $(q, \bar{q}, p, \bar{p})$ for J_+ such that

$$(\Omega + i\Omega_+) = dq \wedge dp \ ,$$

where we choose some polarization. Also we can pick up the Darboux complex coordinates $(Q, \bar{Q}, P, \bar{P})$ for J_- such that

$$(\Omega + i\Omega_-) = dQ \wedge dP \ ,$$

with some polarization. These two choices of coordinates are related to each other by the symplectomorphism for Ω. There exist the coordinates $(q, \bar{q}, P, \bar{P})$ and the generating function $K_\alpha(q, \bar{q}, P, \bar{P})$ such that the corresponding symplectomorphism is defined by the formulas

$$p = \frac{\partial K_\alpha}{\partial q} \ , \qquad \bar{p} = \frac{\partial K_\alpha}{\partial \bar{q}} \ , \qquad Q = \frac{\partial K_\alpha}{\partial P} \ , \qquad \bar{Q} = \frac{\partial K_\alpha}{\partial \bar{P}} \ .$$

Using these expressions we can rewrite the symplectic forms in the new coordinates $(q, \bar{q}, P, \bar{P})$ as

$$\Omega = \frac{1}{2}\frac{\partial^2 K_\alpha}{\partial q \partial P} dq \wedge dP + \frac{1}{2}\frac{\partial^2 K_\alpha}{\partial q \partial \bar{P}} dq \wedge d\bar{P} + \text{c.c.} \ , \tag{4.17}$$

$$\Omega_+ = \frac{i}{2}\frac{\partial^2 K_\alpha}{\partial q \partial \bar{q}} d\bar{q} \wedge dq + \frac{i}{2}\frac{\partial^2 K_\alpha}{\partial \bar{q} \partial \bar{P}} d\bar{q} \wedge d\bar{P} + + \frac{i}{2}\frac{\partial^2 K_\alpha}{\partial \bar{q} \partial P} d\bar{q} \wedge dP - \text{c.c.} \ , \tag{4.18}$$

$$\Omega_- = \frac{i}{2}\frac{\partial^2 K_\alpha}{\partial P \partial \bar{P}} dP \wedge d\bar{P} + \frac{i}{2}\frac{\partial^2 K_\alpha}{\partial \bar{P} \partial q} dq \wedge d\bar{P} + \frac{i}{2}\frac{\partial^2 K_\alpha}{\partial \bar{P} \partial \bar{q}} d\bar{q} \wedge d\bar{P} - \text{c.c.} \tag{4.19}$$

These are local expressions for Ω, $\Omega_\pm$. From them we can easily read off the complex structure and the bihermitian metric. The bihermitian metric can be expressed in terms of the second derivatives of K, although the expression is non-linear. Moreover all formulas depend on the choice of polarization $(q, \bar{q}, P, \bar{P})$. The polarization can be changed and the generating function should be replaced by the appropriate Legendre transform of the original K_α.

4.3 general case

The general case can be thought of as a mixture of two previously considered special cases, the linear and non-linear cases. We will avoid here the full list of explicit formulas since

they are quite lengthy (see [9] for some of the explicit expressions). We just sketch the idea behind their derivation. As we said before the crucial point is that the complex coordinates for J_+ are related to the complex coordinates for J_- through the Poisson diffeomorphism for σ. Let the coordinates (4.11) be

$$(z, \bar{z}, z', \bar{z}', q, \bar{q}, p, \bar{p}) \ , \tag{4.20}$$

where we choose some polarization along σ and the coordinates (4.12)

$$(z, \bar{z}, z', \bar{z}', Q, \bar{Q}, P, \bar{P}) \ , \tag{4.21}$$

with another polarization along σ. The coordinates (4.20) and (4.21) are related to each other by the Poisson diffeomorphism for σ which can be encoded in the generating function $K_\alpha(z, \bar{z}, z', \bar{z}', q, \bar{q}, P, \bar{P})$ (as a generating function it has ambiguities in its definition). Expressing the complex structures $J_\pm$ in the new coordinates $(z, \bar{z}, z', \bar{z}', q, \bar{q}, P, \bar{P})$ through the derivatives of K_α one can show that the integrability conditions (3.6) have a solution for $\omega_\pm$ written in terms of second derivatives of K_α. In the coordinates $(z, \bar{z}, z', \bar{z}', q, \bar{q}, P, \bar{P})$ the bihermitian metric g can be written in terms of second derivatives of K_α. In general the relation will be non-linear in terms of K_α. Namely the second relation in (3.6) is solved locally by

$$\omega_\pm = d(\mathrm{Re}\,\lambda_\pm) + d^c_\pm(\mathrm{Im}\,\lambda_\pm) \ , \tag{4.22}$$

where $\lambda_\pm$ are $(1,0)$-forms with respect to $J_\pm$ complex structures. It should be stressed that in (4.22) we took into account that $\omega_\pm$ are $(1,1)$-forms with respect to the $J_\pm$ complex structures. The first condition in (3.6) implies the following compatibility condition between one forms $\lambda_\pm$

$$d^c_+ d(\mathrm{Re}\,\lambda_+) + d^c_- d(\mathrm{Re}\,\lambda_-) = 0 \ . \tag{4.23}$$

Using the form of the complex structures $J_\pm$ in the coordinates $(z, \bar{z}, z', \bar{z}', q, \bar{q}, P, \bar{P})$ we can resolve this condition as

$$\mathrm{Re}\,\lambda_+ \ = \ \frac{i}{2}(\bar{\partial}_{\bar{P}} + \bar{\partial}_{\bar{z}} + \partial_{z'})K_\alpha - \text{c.c.} \ , \tag{4.24}$$

$$\mathrm{Re}\,\lambda_- \ = \ \frac{i}{2}(\bar{\partial}_{\bar{q}} + \bar{\partial}_{\bar{z}} + \bar{\partial}_{\bar{z}'})K_\alpha - \text{c.c.} \ , \tag{4.25}$$

where it is written up to d-exact terms which disappear in the final expressions for $\omega_\pm$. In the expressions (4.24) and (4.25) we use the locally defined differentials adapted to our coordinates. Now we can read off from (4.22), (4.24) and (4.25) the expression for the bihermitian metric g, which will be in general non-linear in K_α. The locally defined 2-forms $d(\mathrm{Re}\,\lambda_\pm)$ will be non-degenerate if the metric g is non-degenerate. Thus we are dealing with locally defined symplectic structures $d(\mathrm{Re}\,\lambda_\pm)$.

Similar ideas of using Poisson diffeomorphism for σ can be utilized in order to generate new examples of generalized Kähler metrics, see [7], [4].

5 Global issues vs gerbes

In order to understand the global issues we have to figure out how to glue the local formulas discussed in the previous Section. There are number of complications which we are facing. One of them is the dependence of our formulas on the polarization which we have to pick up on the leaf of σ in order to write everything down. The change in the polarization leads to a non-linear Legendre transform of K_α which is unclear how to interpret. The second problem is that we understand only the local description of the generalized Kähler geometry in the neighborhood of the regular point and how one deals with the irregular points is unclear to us.

Below we offer some partial results on the global issues. In the Kähler case the holomorphic line bundles with Hermitian structure play a central role while in the generalized Kähler case the gerbes become important. Gerbes are a geometrical realization of $H^3(M,\mathbb{Z})$ in a manner analogous to the way a line bundle is geometrical realization of $H^2(M,\mathbb{Z})$ [1].

5.1 biholomorphic gerbe

The case when $\sigma = 0$ is relatively simple one. We have to glue together the local expressions (4.15) for $\omega_\pm$. On the double intersection $U_\alpha \cap U_\beta$ we have

$$K_\alpha - K_\beta = f_{\alpha\beta}(z,z') + g_{\alpha\beta}(z,\bar{z}') + \bar{f}_{\alpha\beta}(\bar{z},\bar{z}') + \bar{g}_{\alpha\beta}(\bar{z},z') \ , \tag{5.26}$$

where $f_{\alpha\beta}(z,z')$ is J_+-holomorphic function on $U_\alpha \cap U_\beta$ and $g_{\alpha\beta}(z,\bar{z}')$ is J_--holomorphic function on $U_\alpha \cap U_\beta$. Assuming that $H \in H^3(M,\mathbb{Z})$ we arrive at the following picture involving the gerbes. We can define over any triple intersections the two sets of transition functions

$$G_{\alpha\beta\gamma}(z) \ , F_{\alpha\beta\gamma}(z') \ : \ U_\alpha \cap U_\beta \cap U_\gamma \ , \rightarrow \ \mathbb{C}_* \ , \tag{5.27}$$

which are antisymmetric under permutations of the open sets and satisfy the cocycle condition on the four-fold intersection. Moreover $G_{\alpha\beta\gamma}(z)$ is holomorphic function with respect to both complex structures, $F_{\alpha\beta\gamma}(z')$ is homolorphic for J_+ and anti-holomorphic for J_-. We refer to such G's as biholomorphic gerbes and to F's as twisted biholomorphic gerbes. We impose the following "bihermitian" conditions

$$G_{\alpha\beta\gamma} F^{-1}_{\alpha\beta\gamma} = h^+_{\alpha\beta} h^+_{\beta\gamma} h^+_{\gamma\alpha} \ , \qquad\qquad G_{\alpha\beta\gamma} \bar{F}_{\alpha\beta\gamma} = h^-_{\alpha\beta} h^-_{\beta\gamma} h^-_{\gamma\alpha} \ , \tag{5.28}$$

where $h^\pm_{\alpha\beta}$ are $J_\pm$-holomorphic functions on double intersections. One can easily see that the biholomorphic and twisted biholomorphic gerbes are are both Hermitian if the conditions (5.28) are satisfied. From the conditions (5.28) it follows that there exists real

functions K_α over a patch U_α where

$$h^+_{\alpha\beta}\bar{h}^-_{\alpha\beta}(h^-_{\alpha\beta})^{-1}(\bar{h}^+_{\alpha\beta})^{-1} = e^{K_\alpha}e^{-K_\beta} \ . \tag{5.29}$$

Comparing with the expression (5.30) we have $h^+_{\alpha\beta} = \exp(f_{\alpha\beta})$ and $h^-_{\alpha\beta} = \exp(g_{\alpha\beta})$. The explicit example of this construction is given by the generalized Kähler geometry on $S^3 \times S^1$, see [8].

5.2 general case

Here we can offer only partial result and some observations. If we are dealing with the regular generalized Kähler manifold then we can glue the local expressions for $\omega_\pm$ on the double intersections $U_\alpha \cap U_\beta$ as follows

$$K_\alpha - K_\beta = f_{\alpha\beta}(z, z', q) + g_{\alpha\beta}(z, \bar{z}', P) + \bar{f}_{\alpha\beta}(\bar{z}, \bar{z}', \bar{q}) + \bar{g}_{\alpha\beta}(\bar{z}, z', \bar{P}) \ , \tag{5.30}$$

where we explicitly ignore the issue of polarization. Assuming that $H \in H^3(M, \mathbb{Z})$ and proceeding formally we still arrive at the same notion of the biholomorphic gerbe $G_{\alpha\beta\gamma}(z)$ and the twisted biholomorphic gerbe $F_{\alpha\beta\gamma}(z')$ which we have discussed above. The properties (5.28) and (5.29) are still satisfied, however now $h^+_{\alpha\beta}(z, z', q) = \exp(f_{\alpha\beta})$ is a J_+-holomorphic function of special type (it does not depend on some of the coordinates) and $h^-_{\alpha\beta}(z, \bar{z}', P) = \exp(g_{\alpha\beta})$ is a J_--holomorphic function of special type.

6 Summary

Here we presented a discussion of the local and global aspects of the generalized Kähler geometry. We reviewed the local description in terms of the generalized Kähler potential which is valid in the neighborhood of a regular point. The expression for the bihermitian metric involves the second derivatives of a potential and would be non-linear in general. Thus one can refer to the generalized Kähler geometry as a non-linear generalization of the Kähler geometry. The tools of Poisson geometry are crucial in the derivations of the present results.

There are many open questions which should be addressed. How to extended the local description to the neighborhood of a irregular point? How to properly interpret the choice of polarization which is needed for the construction to work? In particular it is unclear how to deal with the different choices of the polarization while discussing the global issues.

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